Completion of Linear Differential Systems to Involution*

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Abstract. In this paper we generalize the involutive methods and algorithms devised for polynomial ideals to differential ones generated by a finite set of linear differential polynomials in the differential polynomial ring over a zero characteristic differential field. Given a ranking of derivative terms and an involutive division, we formulate the involutivity conditions which form a basis of involutive algorithms. We present an algorithm for computation of a minimal involutive differential basis. Its correctness and termination hold for any constructive and noetherian involutive division. As two important applications we consider posing of an initial value problem for a linear differential system providing uniqueness of its solution and the Lie symmetry analysis of nonlinear differential equations. In particular, this allows to determine the structure of arbitrariness in general solution of linear systems and thereby to find the size of symmetry group.

1 Introduction

Among the properties of systems of analytical partial differential equations (PDEs) which may be investigated without their explicit integration there are compatibility and formulation of an *initial-value problem* providing existence and uniqueness of the solution. The classical Cauchy-Kowalevsky theorem establishes a certain class of quasilinear PDEs which admit posing such an initial-value problem. The main obstacle in investigating other classes of PDE systems of some given order q is existence of *integrability conditions*, that is, such relations for derivatives of order $\leq q$ which are differential but not pure algebraic consequences of equations in the system.

An *involutive* system of PDEs has all the integrability conditions incorporated in it. This means that prolongations of the system do not reveal integrability conditions. Extension of a system by its integrability conditions is called *completion*. The concept of involutivity was invented hundred years ago by E.Cartan [1] in his investigation of the Pfaff type equations in total differentials. For these purposes he used the exterior calculus developed by himself. The Cartan approach was generalized by Kähler[2] to arbitrary systems of exterior differential equations. The underlying completion procedure [3] was implemented in [4,5].

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In his study of the formal power series solutions of PDEs, Riquier introduced [6] a class of relevant *rankings* for partial derivatives and considered systems of *orthonomic* equations which are solved with respect to the highest rank derivatives called *principal*. Thereby, these derivatives, by the equations in the system, are defined in terms of the other derivatives called *parametric*. An integrability condition gives a constraint for parametric derivatives, and that of them of the highest ranking becomes the principal derivative. Recently Riquier's class of rankings was generalized in [7].

Janet made the further development of Riquier's approach. He observed [8] that the integrability conditions may occur only from prolongations with respect to certain independent variables called *nonmultiplicative*. Prolongations with respect to the rest of variables called *multiplicative* never lead to integrability conditions. Given a set of principal derivatives, Janet gave the prescription how to separate variables into multiplicative and nonmultiplicative for every equation in the system. He formulated, on this ground, the *involutivity conditions* for orthonomic systems and designed an algorithm for their completion. This approach to completion is known as *Riquier-Janet theory* and was implemented in [9,10,11].

A system satisfying the Janet involutivity conditions is often called *passive*. This involutivity is generally coordinate dependent. On the other hand, the modern *formal theory* of PDEs developed in 60s-70s by Spencer and others (see [12,13]) allows to formulate the involutivity intrinsically, in a coordinate independent way. The formal theory relies on another definition of multiplicative and nonmultiplicative variables which was known to Janet as long ago as in 20s, but called nowadays after Pommaret because of its importance in the technique presented in [12]. The implementation in Axiom of completion based on the formal theory was presented in [14,15].

Thomas in [16] used another separation of independent variables into multiplicative and nonmultiplicative and generalized the Riquier-Janet theory to non-orthonomic algebraic PDEs. Given a system of PDEs, he showed that in a finite number of steps one can: (i) check its compatibility; (ii) if the system is compatible, then split it into a finite number of *simple* systems involving generally both equations and inequalities and such that their equation parts are orthonomic and can be completed to involution. This splitting is similar to that generated by the Rosenfeld-Gröbner algorithm [17].

In paper [18] for Pommaret separation of independent variables it was shown that involutive (passive) basis of a non-differential polynomial ideal is a Gröbner basis. The implementation in Reduce of the proposed completion algorithm for polynomial bases demonstrated a high computational efficiency of the involutive technique. However, Pommaret bases may not exist for positive dimensional ideals unlike Janet and Thomas bases.

The above classical separations of variables into multiplicative and nonmultiplicative are particular cases of *involutive monomial division*, a concept invented and analyzed in [19] (cf. [20]). The polynomial completion algorithms designed for a general involutive division [19,21] were implemented in

Reduce for Pommaret division. Different involutive divisions and completion of monomial sets have also been implemented in Mathematica [22]. In [23] we generalized the algorithm of paper [21] to arbitrary completion ordering.

One more efficient method for the completion of linear PDEs to an involutive form called *standard* which is not based on the separation of variables was developed in [24] and implemented in Maple. The extension of this method to nonlinear PDEs is given in paper [25].

In the present paper we generalize the involutive methods and algorithms devised in [19,21,23] for polynomial ideals to differential ideals generated by a finite set of linear polynomials. We formulate the involutivity conditions for the differential case. If a set satisfies the involutivity conditions it is called an *involutive basis*. Similar to the pure algebraic case, a linear involutive basis is a differential Gröbner basis [26,27] which is not generally reduced. We present an algorithm for computation of a minimal involutive basis. This algorithm is the straightforward generalization of the polynomial involutive algorithm [21,23]. As well as for the latter, the correctness and termination of the former hold for any constructive and noetherian involutive division.

An important application of the involutive method is posing an initial value problem providing the unique solution of a system of PDEs. For linear involutive systems we formulate such an initial value problem and thereby generalize the classical results of Janet [8] to arbitrary involutive divisions. This formulation makes it possible, among other things, to reveal the structure of arbitrariness in general solution. Given a linear involutive basis, we write also the explicit formulae for the Hilbert function and the Hilbert polynomial of the corresponding differential ideal which are the straightforward generalizations of their polynomial analogues [20,22].

Another important application of the new algorithm is the *Lie symmetry analysis* of nonlinear differential equations. It is because of the fact that completion to involution is the most general and universal method of integrating the determining system of linear PDEs for infinitesimal Lie symmetry generators [28]. Moreover, an involutive form of determining equations allows to construct the Lie symmetry algebra without their explicit integration [29]. In particular, for an involutive determining system the size of symmetry group can easily be found that was shown for Janet bases in [11]. Though reduced Gröbner bases for the determining equations do not generally reveal information on Lie symmetry groups, and more generally on the solution space, so explicitly as involutive bases, they are also very useful for Lie symmetry analysis as shown in [30]. The facilities of the Maple package devised by the first author and used in the paper go far beyond linear differential systems, and it can also be fruitfully applied to nonlinear systems.

2 Preliminaries

Let $\mathbb{R} = \mathbb{K}\{y_1, \ldots, y_m\}$ be a differential polynomial ring [31,32] with the set of differential indeterminates $\{y_1, \ldots, y_m\}$, and $\mathbb{K} \subset \mathbb{R}$ is a differential field

of zero characteristic with a finite number of mutually commuting derivation operators $\partial/\partial x_1, \ldots, \partial/\partial x_n$. Elements in $\mathbb{R}$ are differential polynomials in $\{y_1, \ldots, y_m\}$. In this paper we use the following notations and conventions:

$f, g, h, p \in \mathbb{R}$ are linear differential polynomials.
$F, G, H \subset \mathbb{R}$ are finite sets of linear differential polynomials.
$\mathcal{F} = \{f = 0 \mid f \in F\}$ is a linear system of PDEs.
$\mathbb{N} = \{0, 1, 2, \ldots\}$ is the set of nonnegative integers.
$\alpha, \beta, \gamma \in \mathbb{N}^n$ are multiindices.
$lcm(\alpha, \beta)$ is the least common multiple of $alpha, beta$.
$X = \{x_1, \ldots, x_n\}$ is the set of independent variables.
$R = K[X]$ is the polynomial ring over the field K of zero characteristic.
$R \supset \mathbb{M} = \{x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid \alpha_i \in \mathbb{N}\}$ is the set of monomials in X.
$i = 1, \ldots, n$ indexes derivation operators $\partial_i = \partial/\partial_{x_i}$.
$j = 1, \ldots, m$ indexes indeterminates y_j.
u, v, w are elements in $\mathbb{M}$.
$U, V \subset \mathbb{M}$ are finite monomial sets.
(U) is the monomial ideal in R generated by U.
$deg_i(u)$ is the degree of x_i in $u \in \mathbb{M}$.
$deg(u) = \sum_{i=1}^n deg_i(u)$ is the total degree of u.
$\partial_\alpha y_j = \frac{\partial^{\alpha_1 + \cdots + \alpha_n}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} y_j$ is a derivative.
$ord(\partial_\alpha y_j) = \sum_{i=1}^n \alpha_i$ is the order of $\partial_\alpha y_j$.
θ, ϑ are derivatives.
$lcm(\partial_\alpha y_j, \partial_\beta y_j) = \partial_{lcm(\alpha,\beta)} y_j$.
$\prec$, $\prec_c$ are rankings of derivatives.
$ld(f)$ is the leading derivative in $f \in \mathbb{R}$.
$lc(f) \in K$ is the coefficient of $ld(f)$.
$ld(F)$ is the set of leading derivatives in $F \subset \mathbb{R}$.
$[F]$ is the differential ideal in $\mathbb{R}$ generated by F.
L is an involutive division.
$L(u, U)$ is the set of $(L-)$multiplicative monomials for $u \in U$.
$NM_L(u, U)$ is the set of $L-$nonmultiplicative variables for $u \in U$.
$M_L(u, U)$ is the set of $L-$multiplicative variables for $u \in U$.
$x^\alpha \in \mathbb{M}$ is the monomial associated with the derivative $\partial_\alpha y_j$.
$\cup_{j=1}^m U_j$ is the monomial set associated with the set $ld(F) = \cup_{j=1}^m \{ld_j(F)\}$.
$W = \cup_{j=1}^m \{W_j \mid W_j \subset M\}$ is the complementary set for $\cup_{j=1}^m U_j$.
$\mathcal{G}$ is the set of $L-$generators of W.
$\vartheta = \partial_L \theta$ is a multiplicative prolongation of θ.
$\partial_{x_i} \cdot \theta$ is the nonmultiplicative prolongation of θ w.r.t. x_i.
$\partial_\alpha \cdot \theta$ is a nonmultiplicative prolongation of θ.
$NF_L(p, F)$ is the $L-$normal form of p modulo F.
$NM_L(f, F) \subseteq X$ is the set of $(L-)$nonmultiplicative variables for $f \in F$.
$C_L(F) = \cup_{\theta \in ld(F)} \{\vartheta \mid \vartheta = \partial_L \theta\}$ is the $L-$cone generated by F.

In this paper we distinguish two rankings (c.f. [23]): a *main ranking* and a *completion ranking* denoted by $\succ$ and $\succ_c$, respectively. The main ranking

will be used, as usually, for isolation of the leading derivatives in differential polynomials whereas the completion ranking serves for taking the lowest nonmultiplicative prolongations by the normal strategy [19] and thereby controlling the property of partial involutivity introduced in Sect. 4.

3 Basic Concepts and Definitions

Throughout this paper we exploit the well-known algorithmic similarities between pure algebraic polynomial systems and linear differential systems [13,33]. In so doing, the basic algorithmic ideas go back to Janet [8] who invented the constructive approach to study of PDEs in terms of the corresponding monomial sets which is based on the following association between derivatives and monomials:

$$\partial_\alpha y_j = \frac{\partial^{\alpha_1+\cdots+\alpha_n} y_j}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}} \Longleftrightarrow x^\alpha = x_1^{\alpha_1}\cdots x_n^{\alpha_n}\,. \tag{1}$$

The monomials associated with the different indeterminates y_j are to be considered as belonging to different monomial sets $U_j \in \mathbb{M}$ indexed by subscript j of the indeterminate.

Definition 1. [32] A total ordering $\prec$ over the set of derivatives $\partial_\alpha y_j$ is called a *ranking* if it satisfies: (i) $\partial_i\partial_\alpha y_j \succ \partial_\alpha y_j$, (ii) $\partial_\alpha y_j \succ \partial_\beta y_k \Longleftrightarrow \partial_\gamma\partial_\alpha y_j \succ \partial_\gamma\partial_\beta y_k$ for all $i, j, k, \alpha, \beta, \gamma$. A ranking $\prec$ is said to be *orderly* if $\theta \succ \vartheta$ whenever $ord(\theta) > ord(\vartheta)$.

The association (1) implies the reduction of a ranking $\prec$ to the associated admissible monomial ordering, and throughout the paper we shall assume that

$$\partial_1 \succ \partial_2 \succ \cdots \succ \partial_n \Longleftrightarrow x_1 \succ x_2 \succ \cdots \succ x_n\,. \tag{2}$$

Remark 2. Given a finite set $F \subset \mathbb{R}$ and a ranking $\succ$, set $ld(F)$ of the leading derivatives is partitioned $ld(F) = \cup_j ld_j(F)$ into subsets $ld_j(F)$ corresponding to different indeterminates y_j which occur in $ld(F)$. For an involutive division L defined as follows each subset generates for every its element the separation of independent variables into multiplicative and nonmultiplicative ones.

Definition 3. [19] An *involutive division* L on $\mathbb{M}$ is given, if for any finite monomial set $U \subset \mathbb{M}$ and for any $u \in U$ there is given a submonoid $L(u, U)$ of $\mathbb{M}$ satisfying the conditions:

(a) If $w \in L(u, U)$ and $v|w$, then $v \in L(u, U)$.
(b) If $u, v \in U$ and $uL(u, U) \cap vL(v, U) \neq \emptyset$, then $u \in vL(v, U)$ or $v \in uL(u, U)$.
(c) If $v \in U$ and $v \in uL(u, U)$, then $L(v, U) \subseteq L(u, U)$.
(d) If $V \subseteq U$, then $L(u, U) \subseteq L(u, V)$ for all $u \in V$.

Elements of $L(u,U)$ are called *multiplicative* for u. If $w \in uL(u,U)$, u is called an involutive divisor or $(L-)$*divisor* of w. In such an event the monomial $v = w/u$ is called $L-$*multiplicative* for u. If u is a conventional divisor of w but not $L-$divisor, then v is called *nonmultiplicative* for u.

Remark 4. Definition 3 for every $u \in U$ provides the partition

$$X = M_L(u,U) \cup NM_L(u,U), \quad M_L \subset L(u,U) \tag{3}$$

of the set of variables $X = \{x_1, \ldots, x_n\}$ into subset $M_L(u,U)$ of *multiplicative variables* for u and subset $NM_L(u,U)$ of the remaining *nonmultiplicative variables*. Conversely, if for any finite set $U \subset \mathbb{M}$ and any $u \in U$ the partition (3) of variables into multiplicative and nonmultiplicative is given such that the corresponding submonoid $L(u)$ satisfies the conditions (b)-(d) in Definition 3, then the partition generates an involutive division.

Definition 5. [19] A monomial set U is called $L-$*autoreduced* if $uL(u,U) \cap vL(v,U) = \emptyset$ holds for all distinct $u, v \in U$.

Definition 6. [19] A monomial set $\tilde{U}$ is called an $L-$*completion* of a set $U \subseteq \tilde{U}$ if

$$(\forall u \in U)\ (\forall w \in \mathbb{M})\ (\exists v \in \tilde{U})\ \ [\ uw \in vL(v,\tilde{U})\ ]\,.$$

If there exists a finite $L-$completion $\tilde{U}$ of a finite set U, then the latter is called *finitely generated* with respect to L. The involutive division L is *noetherian* if every finite set U is finitely generated with respect to L. If $\tilde{U} = U$, then U is called $L-$*complete*. An $L-$autoreduced and complete set is called $(L-)$*involutive*.

Definition 7. [19] Given a monomial set U, the set $\cup_{u\in U}\, u\,\mathbb{M}$ is called *the cone* generated by U and denoted by $C(U)$. The set $\cup_{u\in U}\, u\, L(u,U)$ is called *the involutive cone* of U with respect to L and denoted by $C_L(U)$.

Thus, the set $\tilde{U}$ is an $L-$completion of U if $C(\tilde{U}) = C_L(\tilde{U}) = C(U)$. Correspondingly, for an involutive set U the equality $C(U) = C_L(U)$ holds.

Whereas noetherity provides existence of a finite involutive basis for any polynomial ideal, another important properties of an involutive division called *continuity* and *constructivity* provide the algorithmic construction of involutive bases [19]. Continuity implies involutivity when the local involutivity holds whereas constructivity strengthens continuity and allows to compute involutive bases by sequential examination of single nonmultiplicative prolongations only. We refer to papers [19,21,23] for description of these topics in detail. In those papers some examples of involutive divisions were studied (see also [34]) which include three divisions called after Janet, Thomas and

Pommaret, because they have used the corresponding separations of variables for involutivity analysis of PDEs [8,16,12]. Other two divisions called Division I and II were introduced in [21], and a class of involutive divisions called Induced division, since every division in the class is induced by an admissible monomial orderings, was introduced in [23]. All those divisions are constructive and, except Pommaret division, they are noetherian. Below we use three of those divisions defined as follows.

Definition 8. Janet division [8]. Let $U \subset \mathbb{M}$ be a finite set. Divide U into groups labeled by non-negative integers $\alpha_1, \ldots, \alpha_i$ $(1 \leq i \leq n)$:

$$[\alpha_1, \ldots, \alpha_i] = \{\ u\ \in U \mid \alpha_j = deg_j(u),\ 1 \leq j \leq i\ \}.$$

Then x_i is multiplicative for $u \in U$ if $i = 1$ and $deg_1(u) = \max\{deg_1(v) \mid v \in U\}$, or $u \in [\alpha_1, \ldots, \alpha_{i-1}]$ and $deg_i(u) = \max\{deg_i(v) \mid v \in [\alpha_1, \ldots, \alpha_{i-1}]\}$ for $i > 1$.

Definition 9. Pommaret division [12]. For a monomial $u = x_1^{\alpha_1} \cdots x_k^{\alpha_k}$ with $\alpha_k > 0$ the variables $x_j, j \geq k$ are considered as multiplicative and the other variables as nonmultiplicative. For $u = 1$ all the variables are multiplicative.

Definition 10. Lexicographically induced division [23]. A variable x_i is nonmultiplicative for $u \in U$ if there is $v \in U$ such that $v \prec_{Lex} u$ and $deg_i(u) < deg_i(v)$, where $\succ_{Lex}$ denotes the lexicographical ordering.

In the sequel Janet, Pommaret and Lexicographically induced divisions will be distinguished by the subscripts J, P and D_{Lex}, respectively.

Example 11. Separation of variables for set $U = \{x_1^2x_3, x_1x_2, x_1x_3^2\}$ and ordering (2) for the above defined three divisions:

Element in U	Separation of variables					
	Janet		Pommaret		Lex. induced	
	M_J	NM_J	M_P	NM_P	$M_{D_{Lex}}$	$NM_{D_{Lex}}$
$x_1^2x_3$	x_1, x_2, x_3	$-$	x_3	x_1, x_2	x_1	x_2, x_3
x_1x_2	x_2, x_3	x_1	x_2, x_3	x_1	x_1, x_2	x_3
$x_1x_3^2$	x_3	x_1, x_2	x_3	x_1, x_2	x_1, x_2, x_3	$-$

The corresponding L-completions of U are

$$\begin{aligned}
&\tilde{U}_J = \{x_1^2x_3, x_1x_2, x_1x_3^2, x_1^2x_2\},\\
&\tilde{U}_P = \{x_1^2x_3, x_1x_2, x_1x_3^2, x_1^2x_2, \ldots, x_1^{i+2}x_2, \ldots, x_1^{j+2}x_3, \ldots\},\\
&\tilde{U}_{D_{Lex}} = \{x_1^2x_3, x_1x_2, x_1x_3^2, x_1x_2x_3\}.
\end{aligned}$$

where $i, j \in \mathbb{N}$. This example explicitly shows the non-noetherity of Pommaret division.

Definition 12. Given a finite set $F \subset \mathbb{R}$, a ranking $\succ$ and an involutive division L, the derivative $\vartheta = \partial_\beta y_j$ will be called a *multiplicative prolongation* of $\theta = \partial_\alpha y_j \in ld_j(F)$ and denoted by $\vartheta = \partial_L \theta$, if the associated monomials satisfy $x^\beta \in x^\alpha L(x^\alpha, U_j)$. Otherwise the prolongation will be called *nonmultiplicative*. Respectively, the corresponding prolongation $\partial_\beta f$ of the element $f \in F$ with $ld(f) = \partial_\alpha y_j$ will be called *multiplicative* and denoted by $\partial_L(f)$ or *nonmultiplicative*. The set $C_L(F) = \cup_{\theta \in ld(F)} \{\vartheta \mid \vartheta = \partial_L \theta\}$ will be called the $L-$*cone* generated by F. If $\partial_i f$ is a nonmultiplicative prolongation of $f \in F$, we shall write $x_i \in NM_L(f, F)$.

4 Linear Involutive Differential Bases

In this section we generalize the results obtained in papers [19,23] for commutative algebra to differential algebra of linear polynomials. Proofs of the theorems are omitted because of similarity with the proofs of their algebraic analogues.

Definition 13. Given an involutive division L, a finite set $F \subset \mathbb{R}$ of linear differential polynomials, a ranking $\succ$ and a linear polynomial $p \in \mathbb{R}$, we shall say:

1. p is $L-$*reducible modulo* $f \in F$ if p has a term $a\,\theta$, $(a \in K \setminus \{0\})$ such that $\theta = \partial_L ld(f)$. It yields the $L-$*reduction* $p \to g = p - (a/lc(f))\,\partial_\beta f$ where $\partial_\beta ld(f) = \theta$.
2. p is $L-$*reducible modulo* F if there is $f \in F$ such that p is $L-$reducible modulo f.
3. p is *in* $L-$*normal form modulo* F, if p is not $L-$reducible modulo F.

We denote the $L-$normal form of p modulo F by $NF_L(p, F)$.

As a $L-$normal form algorithm one can use the following differential analogue of the polynomial normal form algorithm [19]:

Algorithm **InvolutiveNormalForm:**

Input: p, F, L, $\prec$
Output: $h = NF_L(p, F)$
begin
  $h := p$
  while exist $f \in F$ and a term $a\,\theta$ $(a \in \mathbb{K} \setminus \{0\})$ of h
    such that $\theta = \partial_L ld(f)$ **do**
    choose the first such f
    $h := h - (a/lc(f))\,\partial_\beta f$ where $\partial_\beta ld(f) = \theta$
  end
end

Correctness and termination of this algorithm is an obvious consequence of Definition 13 and correctness and termination of the polynomial $L-$normal form algorithm [19].

Definition 14. A finite set F is called $L-$*autoreduced* if every $f \in F$ is irreducible modulo any other element $g \in F$. An $L-$autoreduced set F is called $(L-)$*involutive* if

$$(\forall f \in F)\ (\forall \alpha \in \mathbb{N}^n)\ [\ NF_L(\partial_\alpha f, F) = 0\ ].$$

Given a derivative ϑ and an $L-$autoreduced set F, if there exist $f \in F$ such that $ld(f) \prec_c \vartheta$ and

$$(\forall f \in F)\ (\forall \alpha \in \mathbb{N}^n)\ \ (\partial_\alpha ld(f) \prec_c \vartheta)\ \ \ [\ NF_L(\partial_\alpha f, F) = 0\ ]\,, \tag{4}$$

then F is called *partially involutive up to the derivative* ϑ with respect to the ranking $\prec_c$. F is still said to be partially involutive up to ϑ if $\vartheta \prec_c ld(f)$ for all $f \in F$.

Corollary 15. *If $F \subset \mathbb{R}$ is an $L-$involutive set, then every monomial set $U_j \in \mathbb{M}$ $(1 \leq j \leq m)$ associated with $ld_j(F)$ is $L-$involutive.*

Proof. It follows immediately from Definitions 6 and 14. □

Theorem 16. *An $L-$autoreduced set $F \subset \mathbb{R}$ is involutive with respect to a continuous involutive division L iff the following (local) involutivity conditions hold*

$$(\forall f \in F)\ \ (\forall x_i \in NM_L(f, F))\ \ \ [\ NF_L(\partial_{x_i} \cdot f, F) = 0\ ]\,.$$

Correspondingly, partial involutivity (4) holds iff

$$(\forall f \in F)\ (\forall x_i \in NM_L(f, F))\ (\partial_{x_i} \cdot ld(f) \prec_c \vartheta)\ \ [\ NF_L(\partial_{x_i} \cdot f, F) = 0\ ]\,.$$

Theorem 17. *If $F \subset \mathbb{R}$ is an $L-$involutive basis of $[F]$, then it is also a differential Gröbner basis.*

The following theorem and corollary give an involutive analogue of Buchberger chain criterion [35] in application to linear differential bases.

Theorem 18. *Let F be a finite $L-$autoreduced set of linear differential polynomials with respect to a continuous involutive division L, and $NF_L(p, F)$ be an algorithm of $L-$normal form. Then the following are equivalent:*

1. *F is an $L-$involutive differential basis of $[F]$.*
2. *For all $g \in F, x \in NM_L(g,F)$ there is $f \in F$ satisfying $\partial_x \cdot ld(g) = \partial_L ld(f)$ and a chain of elements in F of the form*

$$f \equiv f_k, f_{k-1}, \ldots, f_0, g_0, \ldots, g_{m-1}, g_m \equiv g$$

such that

$$NF_L\left(S_L(f_{i-1}, f_i), F\right) = NF_L\left(S(f_0, g_0), F\right) = NF_L\left(S_L(g_{j-1}, g_j), F\right) = 0$$

where $0 \leq i \leq k$ and $0 \leq j \leq m$, $S(f_0, g_0)$ is the conventional differential S-polynomial [27] and $S_L(f_i, f_j) = \partial_x \cdot f_i - \partial_L f_j$ is its special form which occurs in involutive algorithms.

Corollary 19. *Let F be a finite $L-$autoreduced set, and let $\partial_x \cdot g$ be a non-multiplicative prolongation of $g \in F$. If the following holds*

$$(\forall h \in F)\ (\forall\, \partial_\alpha)\ \ (\,\partial_\alpha ld(h) \cdot u \prec_c ld(g \cdot x)\,)\ \ \ [\ NF_L(h \cdot u, F) = 0\ ]\,,$$

$$(\exists f, f_0, g_0 \in F) \begin{bmatrix} ld(f) = \partial_\beta ld(f_0)\,,\ ld(g) = \partial_\gamma ld(g_0) \\ \partial_x \cdot ld(g) = \partial_L ld(f)\,,\ lcm\,(ld(f_0), ld(g_0)) \prec_c \partial_x \cdot ld(g) \\ NF_L\big(\partial_\beta \cdot f_0, F\big) = NF_L\big(\partial_\gamma \cdot g_0, F\big) = 0 \end{bmatrix}\,,$$

then the prolongation $\partial_x \cdot g$ may be discarded in the course of an involutive algorithm.

5 Completion Algorithm

The below given algorithm **MinimalLinearInvolutiveBasis** is a differential analogue of the polynomial algorithm **MinimalInvolutiveBasis** of paper [23]. In so doing, the conventional (non-involutive) autoreduction which is performed in line 2 of the latter algorithm omitted, as this autoreduction is optional [23].

Validity of the involutive chain criterion used in lines 11 and 23 is provided by Theorem 18 and Corollary 19. The proof of correctness and termination of the differential algorithm is identical to the proof for its polynomial analogue [21,23]. It follows, that if the main ranking $\succ$ is orderly, then, given a generating set of linear differential polynomials and a constructive involutive division, algorithm **MinimalLinearInvolutiveBasis** computes a minimal differential basis whenever the latter exists. If the division is noetherian, the basis is computed for any main ranking.

Though the output basis for a noetherian division does not depend on the completion ranking, the proper choice of the latter may increase efficiency of computation.

Remark 20. If the algorithm **MinimalLinearInvolutiveBasis** takes a conventional differential Gröbner basis of the ideal $[F]$ as an input, then it produces the minimal involutive differential basis just by enlargement of the input set with its irreducible nonmultiplicative prolongations if any. This enlargement is done in the lower **while**-loop.

Algorithm **MinimalLinearInvolutiveBasis**

Input: F, L, $\succ$ (main ranking), $\succ_c$ (completion ranking)
Output: G, a minimal involutive basis of $[F]$

begin
  choose $g \in F$ with the lowest $ld(g)$ w.r.t. $\prec$
  $T := \{(g, ld(g), \emptyset)\};\ \ Q := \emptyset;\ \ G := \{g\}$
  for each $f \in F \setminus \{g\}$ **do**
    $Q := Q \cup \{(f, ld(f), \emptyset)\}$
  repeat
    $h := 0$
    while $Q \neq \emptyset$ **and** $h = 0$ **do**
      choose g in $(g, \theta, P) \in Q$ with the lowest $ld(g)$ w.r.t. $\prec$
      $Q := Q \setminus \{(g, \theta, P)\}$
      if $Criterion(g, \theta, T)$ is false **then** $h := NF_L(g, G)$
    if $h \neq 0$ **then** $G := G \cup \{h\}$
      if $ld(h) = ld(g)$ **then** $T := T \cup \{(h, \theta, P \cap NM_L(h, G))\}$
      else $T := T \cup \{(h, ld(h), \emptyset)\}$
        for each f in $(f, \vartheta, S) \in T$ s.t. $ld(f) \succ ld(h)$ **do**
          $T := T \setminus \{(f, \vartheta, S)\};\ \ Q := Q \cup \{(f, \vartheta, S)\};\ \ G := G \setminus \{f\}$
        for each $(f, \vartheta, S) \in T$ **do**
          $T := T \setminus \{(f, \vartheta, S)\} \cup \{(f, \vartheta, S \cap NM_L(f, G))\}$
    while exist $(g, \theta, P) \in T$ and $x \in NM_L(g, G) \setminus P$ and, if $Q \neq \emptyset$,
      s.t. $ld(\partial_x \cdot g) \prec ld(f)$ for all f in $(f, \vartheta, S) \in Q$ **do**
      choose such $(g, \theta, P), x$ with the lowest $ld(\partial_x \cdot g)$ w.r.t. $\prec_c$
        $T := T \setminus \{(g, \theta, P)\} \cup \{(g, \theta, P \cup \{x\})\}$
      if $Criterion(\partial_x \cdot g, \theta, T)$ is false **then** $h := NF_L(\partial_x \cdot g, G)$
        if $h \neq 0$ **then** $G := G \cup \{h\}$
          if $ld(h) = ld(\partial_x \cdot g)$ **then** $T := T \cup \{(h, \theta, \emptyset)\}$
          else $T := T \cup \{(h, ld(h), \emptyset)\}$
            for each f in $(f, \vartheta, S) \in T$ with $ld(f) \succ ld(h)$ **do**
              $T := T \setminus \{(f, \vartheta, S)\};\ \ Q := Q \cup \{(f, \vartheta, S\});\ \ G := G \setminus \{f\}$
            for each $(f, \vartheta, S) \in T$ **do**
              $T := T \setminus \{(f, \vartheta, S)\} \cup \{(f, \vartheta, S \cap NM_L(f, G))\}$
  until $Q \neq \emptyset$
end

$Criterion(g, \theta, T)$ is true if there is $(f, \vartheta, S) \in T$ such that $ld(g) = \partial_L ld(f)$ and $lcm(\theta, \vartheta) \prec_c ld(g)$.

Example 21. [8] The well-known Janet example with three independent and one dependent variables ($n = 3, m = 1$):

$$\begin{cases} \partial_{11} y - x_2 \partial_{33} y = 0\,, \\ \partial_{22} y = 0\,. \end{cases}$$

The above completion algorithm applied for Janet, Pommaret and Lexicographically induced divisions gives the following involutive bases, which coincide for both pure lexicographical and graded lexicographical main rankings compatible with (2) and which sorted in the descending lexicographical order:

Gröbner basis	Involutive Bases	
	Janet & Pommaret	Lex. Induced
$\partial_{11}y - x_2\partial_{33}y$	$\partial_{11}y - x_2\partial_{33}y$	$\partial_{112}y - \partial_{33}y$
$\partial_{22}y$	$\partial_{122}y$	$\partial_{11333}y$
$\partial_{233}y$	$\partial_{1233}y$	$\partial_{1133}y$
$\partial_{3333}y$	$\partial_{13333}y$	$\partial_{113}y - x_2\partial_{333}y$
	$\partial_{22}y$	$\partial_{11}y - x_2\partial_{33}y$
	$\partial_{233}y$	$\partial_{223}y$
	$\partial_{3333}y$	$\partial_{22}y$
		$\partial_{2333}y$
		$\partial_{233}y$
		$\partial_{3333}y$

The first column contains the reduced differential Gröbner basis, and Janet and Pommaret bases are identical for this example.

6 Initial Value Problem

The results of this section generalize to arbitrary $L-$involutive linear systems those obtained in Riquier-Janet theory [6,8,16], for Janet and Thomas divisions, as well as in the formal theory [12,13] for Pommaret division, on posing an initial value problem providing uniqueness and existence of solutions.

Definition 22. [6,8] If $\theta \in ld(F)$ is a leading derivative in $F \subset \mathbb{R}$, then $\partial_\alpha \theta$ is called a *principal derivative*. A derivative which is not principal is called *parametric*. The monomial set $W = \{\cup_{j=1}^m W_j \mid W_j \subset \mathbb{M}\}$ associated by (1) with the set of parametric derivatives is called a *complementary set* of F.

Proposition 23. *Given a ranking $\prec$, if set F is a linear $L-$involutive basis of differential ideal $[F]$, then the sets of principal and parametric derivatives (complementary set) related to F depend only on $[F]$ and $\prec$ and do not depend on the choice of involutive division L.*

Proof. It follows immediately from the fact that any involutive basis is a Gröbner basis (Theorem 17). □

Lemma 24. (*decomposition lemma*) *Given a noetherian division L and $L-$involutive set $F \subset \mathbb{R}$, every subset W_j in the complementary monomial set of F related to j-th differential indeterminate y_j $(1 \leq j \leq m)$ can be decomposed as a disjoint union*

$$W_j = \cup_{v \in V_j} vL_v, \quad L_v \subseteq L(v, U_j \cup \{v\}), \tag{5}$$

where U_j is the $L-$involutive monomial set (not necessarily nonempty) associated with $ld_j(F)$, and $V_j \in \mathbb{M}$ is a finite subset.

Proof. Let U and W be a pair of monomial sets associated with the principal and parametric derivatives of a differential indeterminate in F. The complementary set W can be written as a disjoint union [36]

$$W = W_0 \cup W_1 \cup \cdots \cup W_d \tag{6}$$

where d is the dimension of monomial ideal (U), W_0 is a finite set, and every W_i $(1 \leq i \leq d)$ is a finite disjoint union[1]

$$W_r = W_{r_1} \cup W_{r_2} \cup \cdots \cup W_{r_k} \tag{7}$$

with

$$W_{r_s} = \{\ w_{r_s} x_{i_{s_1}}^{\alpha_1} \cdots x_{i_{s_r}}^{\alpha_r} \mid \alpha_t \in \mathbb{N},\ 1 \leq t \leq r\ \} \quad (1 \leq s \leq k). \tag{8}$$

For every $v \in W_0$ we shall take $L_v = \{1\}$ in (5). Thus, for $d = 0$ the decomposition (5) $W = W_0 = \cup_{v \in W_0}\{v\}$ holds trivially. If $d > 0$ we consider the finite set

$$V = W_0 \cup_{r=1}^{d} \cup_{s=1}^{k} \{w_{r_s}\}, \tag{9}$$

where monomials w_{r_s} generate W_{r_s} in accordance with (8).

We claim that elements in set (9), and the decompositions (6), (7) they determine can be written such that the union in $W = \cup_{v \in V} vL_v$ with $L_v \subseteq L(v, U \cup \{v\})$ is disjoint in accordance with (5). To prove the claim we define the degree q of set U as $q = \max\{deg(u) \mid u \in U\}$, and choose all the monomials w_{r_s} generating W_{r_s} in (8) such that $deg(w_{r_s}) = q$. Obviously this can always be done by appropriate choice of W_0. Let now V_1 be the set $V_1 = \cup_{r=1}^{d} \cup_{s=1}^{k} \{w_{r_s}\}$, and let $\hat{U}$ be a finite $L-$autoreduced completion of $U \cup V$. The existence of $\hat{U}$ is guaranteed by noetherity of L. Now consider the set $\hat{V} = \hat{U} \cap W \supseteq V_1$. Its $L-$involutivity and property (d) of L in Definition 10 imply

$$(\forall w \in W \setminus W_0)\ (\exists v \in \hat{V})\ \ [\ w \in vL(v, T) \subseteq vL(v, U \cup \{v\})\ ].$$

Thus, we obtain the desired decomposition $W = W_0 \cup_{v \in \hat{V}} vL(v, \hat{U})$. Disjointedness of this union follows from that in (7) and Definition 5 of $L-$autoreduction. This proves the claim and the lemma. □

[1] The union in (7) considered in [36] is not necessarily disjoint. However, unions in (6) and (7) apparently can be rewritten as disjoint by appropriate choice of W_0 and components of W_r.

Definition 25. Those elements v_{j_k} (parametric derivatives) which, in accordance with (5), generate the whole complementary set W, will be called $L-$*generators* of the set. The multiplicative variables x_i satisfying $x_i \in L_{j_k}$ will be called $(L-)$*multipliers* of the generator v_{j_k} and the remaining variables will be called its $(L-)$*nonmultipliers*. The whole set of $L-$generators of W will be denoted by $\mathcal{G}_L$, and in accordance with (5)

$$\mathcal{G}_L = \cup_{j=1}^m V_j \,. \tag{10}$$

For a non-noetherian division L a complementary set may not have a finite set of $L-$generators as the following example shows.

Example 26. Let involutive division L be defined on $\mathbb{M}$ as follows. Variables $x_1, \ldots, x_{n-1}$ are separated into multiplicative and nonmultiplicative by Definition 8. Let the variable x_n be also separated by Definition 8 if $deg_n(u) = 0$ and $u \neq 1$, whereas if $deg_n(u) > 0$ or if $u = 1$, x_n be nonmultiplicative for u. Then, the monomial set $U = \{x_1^2, x_1x_2, x_2\}$ is $L-$involutive in $K[x_1, x_2, x_3]$. Its complementary set has the infinite set of $L-$generators: $\mathcal{G}_L = \{1\} \cup \{x_1\} \cup_{i=1}^{\infty} \{x_3^i\}$.

Remark 27. Decomposition (5) and the underlying $L-$generator set (25) are not uniquely defined, and usually a more compact set $\mathcal{G}$ of $L-$generators (with less number of elements) than that constructed in the proof of Lemma 24 can be chosen. For example, for a Janet basis, $\mathcal{G}_P$ can always be chosen [8] as union (10) of sets V_j such that

$$(\forall\, V_j)\ (\forall v \in V_j)\ [\ L_v = J(v, U_j \cup \{v\}\ ]\quad (1 \leq j \leq m), \tag{11}$$

where J stands for the Janet set of multiplicative monomials. Since for $\hat{U}_j$, as it constructed in the proof, the inclusion $U_j \cup \{v\} \subseteq \hat{U}_j$ holds, the property (d) in Definition 3 implies $J(v, \hat{U}_j) \subset J(v, U_j \cup \{v\})$. Therefore, the set of Janet generators defined by (11) is a subset of that constructed in the proof of Lemma 24.

For a Pommaret basis in the formal theory [12] decomposition (5) is taken in the form

$$W = W_0 \cup_{\{v \in W\ \mid\ deq(v)=q\}} vP(v), \tag{12}$$

where $P(v)$ denotes the set of Pommaret multiplicative monomials for v, and q, as in the proof, is the degree of the basis. The number of Pommaret generators in (12) with i multipliers is called the ith *Cartan character*[2] $(1 \leq i \leq n)$ of the basis and will be denoted by σ_q^i.

[2] Cartan introduced these numbers in his analysis of exterior PDEs [3] and called them *characters*.

Example 28. The complementary set of the monomial ideal (U) for $U = \{x_1^2x_3, x_1x_2, x_1x_3^2\}$ in Example 11 is $W = \cup\{x_1^{i+1} \mid i \in \mathbb{N}\} \cup \{x_2^j x_3^k \mid j,k \in \mathbb{N}\}$. Its most compact sets $\mathcal{G}_J$ and $\mathcal{G}_{D_{Lex}}$ together with their multipliers are:

Janet division		Lex. induced division	
Generator	Multipliers	Generator	Multipliers
1	x_2, x_3	1	x_2, x_3
x_1	$-$	x_1	x_1
x_1^2	x_1	$-$	$-$

We note that if the involutive bases $\tilde{U}_P, \tilde{U}_{D_{Lex}}$ given in Example 11 are sequentially enlarged with every single generator, then the sets of Janet multipliers, in accordance with Remark 2, coincide with the sets of multiplicative variables

$$M_J(1, \tilde{U}_P \cup \{1\}) = \{x_2, x_3\},\ M_J(x_1, \tilde{U}_P \cup \{1\}) = \emptyset,\ M_J(x_1^2, \tilde{U}_P \cup \{x_1^2\}) = \{x_1\}$$

whereas for lexicographically induced division, every set of multipliers is the proper subset of multiplicative variables

$$M_{D_{Lex}}(1, \tilde{U}_{D_{Lex}} \cup \{1\}) = M_{D_{Lex}}(x_1, \tilde{U}_{D_{Lex}} \cup \{x_1\}) = \{x_1, x_2, x_3\}.$$

Theorem 29. (*uniqueness theorem*) *Let $\mathcal{F}$ be an $L-$involutive system of linear PDEs for an orderly ranking. Then $\mathcal{F}$ has at most one solution satisfying the following initial conditions: the derivatives associated with $L-$generators of the complementary monomials are arbitrary functions of their multipliers at the fixed values of their nonmultipliers from coordinates of the initial point $x_i = x_i^{\circ}$ $(1 \leq i \leq n)$), whereas the generators without multipliers are considered to be arbitrary constants.*

Proof. Involutivity of $\mathcal{F}$ with respect to an orderly ranking implies that the associated complementary monomial set contains all the monomials associated with the parametric derivatives. This statement is an immediate consequence of the well-known fact [36] that for a graded monomial ordering the Hilbert function of a polynomial ideal is defined by the monomial ideal generated by the leading monomials of a Gröbner basis of the polynomial ideal.

Furthermore, by association (1), the decomposition (6) yields that every parametric derivative associated with a monomial in $W \setminus W_0$ is produced by differentiation of the uniquely defined parametric derivative ($L-$generator) with respect to its multipliers. Assigning the fixed values to all these parametric derivatives is obviously equivalent to fixing some function of the multipliers. Therefore, given initial point $x_i = x_i^{\circ}$, in addition to the set of arbitrary constants which associated with elements in W_0, all the parametric arbitrariness is determined by functions corresponding to the $L-$generators and which are arbitrary functions of the multipliers at the fixed values of nonmultipliers from coordinates of the initial point. □

Remark 30. Different involutive divisions give obviously equivalent forms of initial value problem providing the uniqueness of solutions. However, given a system of PDEs with an infinite set of parametric derivatives, the writingf of such initial conditions in accordance with Theorem 29 may be more compact for one division than for another. We demonstrate this fact by examples given below.

Theorem 31. (*existence theorem*) *Let $\mathcal{F}$ be an $L-$involutive linear system for an orderly ranking, and let its coefficients be analytic functions in an initial point $(x_i = x_i^{\circ})$. Then $\mathcal{F}$ has precisely one solution which is analytic in this point if all arbitrary functions in the initial data specified in Theorem 29 are analytic in their arguments taking values from coordinates of the initial point.*

Proof. This is identical to the existence proof in Riquier-Janet theory [6,8] (see also [31]). □

Example 32. The complementary monomial set for Janet system in Example 21 is finite and consists of 12 elements

$$W = \{1, x_1, x_2, x_3, x_1x_2, x_1x_3, x_2x_3, x_3^2, x_1x_2x_3, x_1x_3^2, x_3^3, x_1x_3^3\}.$$

By Theorem 31, its general solution depends on 12 arbitrary constants.

Example 33. [12] The system of the first order PDEs with four independent and one dependent variables $(n = 4, m = 1)$ and its completion to involution for Janet or Pommaret division for any ranking compatible with (2) are given by

$$\begin{cases} \partial_1 y + x_2\partial_3 y + y = 0, \\ \partial_2 y + x_1\partial_4 y = 0, \end{cases} \quad \stackrel{J,P-\text{completion}}{\Longrightarrow} \quad \begin{cases} \partial_1 y + x_2\partial_4 y + y = 0, \\ \partial_2 y + x_1\partial_4 y = 0, \\ \partial_3 y - \partial_4 y = 0. \end{cases}$$

The parametric derivatives $\partial_4^i y$ $(i \in \mathbb{N})$ have the only Janet generator $y \Longleftrightarrow 1$ with the only multiplier x_4. Hence, the initial data providing the unique analytic solution are $y|_{x_1=x_1^{\circ},x_2=x_2^{\circ},x_3=x_3^{\circ}} = \phi(x_4)$ with arbitrary function $\phi(x_4)$, analytic at $x_4 = x_4^{\circ}$. The system can explicitly be integrated and its general solution is

$$y = e^{-(x_4-x_4^{\circ})}\phi(x_3 + x_4 - x_1x_2 + x_1^{\circ}x_2^{\circ} - x_3^{\circ}).$$

The Pommaret generators are y and $\partial_4 y$ without multipliers and with multiplier x_4, respectively. This leads to the initial value problem

$$y|_{x_1=x_1^{\circ},x_2=x_2^{\circ},x_3=x_3^{\circ}} = c, \quad \partial_4 y|_{x_1=x_1^{\circ},x_2=x_2^{\circ},x_3=x_3^{\circ}} = \psi(x_4)$$

with arbitrary constant c and arbitrary function ψ. This shows that the Janet initial conditions are written in a more compact form than those of Pommaret.

Example 34. [37] The well-known Lewy example with $n = 3, m = 2$ and $\eta_1, \eta_2 \in \mathbb{K}$

$$\begin{cases} \partial_1 y_1 - 2x_3\, \partial_2 y_1 - \partial_3 y_2 - 2x_1\, \partial_2 y_2 = \eta_1(x_1, x_2, x_3), \\ \partial_1 y_2 + 2x_1\, \partial_1 y_1 + \partial_3 y_1 - 2x_3\, \partial_2 y_2 = \eta_2(x_1, x_2, x_3). \end{cases}$$

This system is involutive for any of Janet, Pommaret or lexicographically induced divisions and the orderly ranking with $\partial_1 y_j \succ \partial_2 y_j \succ \partial_3 y_j,\ y_1 \succ y_2$. Janet generators are y_1, y_2. Each of them has multipliers x_2, x_3. This implies the initial data providing the uniqueness: $y_j|_{x_1 = x_1^\circ} = \phi_j(x_2, x_3)\ (j = 1, 2)$ with arbitrary functions $\phi_j(x_2, x_3)$. Pommaret and lexicographically induced divisions lead to a less compact writing of these conditions.

Remark 35. As shown by Lewy [37] for Example 34, there exist the C^∞ functions η_1, η_2 such that the system has no C^∞ (and even C^1) solutions. Therefore, analyticity in the Theorem 31 statement can not be replaced by smoothness.

We conclude this section with explicit formulae for the Hilbert function $HF_{[F]}$ and Hilbert polynomial $HP_{[F]}$ of differential ideal $[F]$ represented by its linear involutive basis F. These formulae are valid for any involutive division and an orderly ranking. For ordinary differential ideals, that is, for the case of single differential indeterminate ($m = 1$), by association (1), they are the same as in commutative algebra [20,22]. For partial differential case they involve the number m of differential indeterminates

$$HF_{[F]}(s) = m \binom{n+s}{s} - \sum_{j=1}^{m} \sum_{i=0}^{s} \sum_{u \in U_j} \binom{i - deg(u) + \mu(u) - 1}{\mu(u) - 1}, \quad (13)$$

$$HP_{[F]}(s) = m \binom{n+s}{s} - \sum_{j=1}^{m} \sum_{u \in U_j} \binom{s - deg(u) + \mu(u)}{\mu(u)}. \quad (14)$$

Here n is the number of independent variables, U_j is the monomial set associated with the set of leading derivatives $ld_j(F)$, and $\mu(u)$ is the number of multiplicative elements of u.

The first term in the right hand side of (13) is the total number of derivatives of order $\le s$. The triple sum counts the number of principal derivatives among them in accordance with Definition 14 which says that any principal derivative is uniquely obtained by the multiplicative prolongation of one of the leading derivatives in F. Thus, (13) gives the number of parametric derivatives of order $\le s$, and for s large enough it becomes polynomial (14).

In the formal theory [12,13] the Janet formula is used:

$$HP_{[F]} = \sum_{i=1}^{n} \binom{s - q + i - 1}{i - 1} \sigma_q^i.$$

Here the Hilbert polynomial [8] is written in terms of Cartan characters σ_q^i (see Remark 2). Apparently, this is (14), rewritten for Pommaret division in terms of Cartan characters.

7 Lie Symmetry Analysis of PDEs

Lie symmetry methods and their computerization yield a powerful practical tool for analysis of nonlinear differential equations (see the review article [28] and references therein for more details). We present here the basic computational formulae and demonstrate, by two simple examples with a single nonlinear evolution equation, application of the above described involutive methods to finding the classical infinitesimal symmetries.
Given a finite system of polynomial-nonlinear PDEs

$$f_k(x_i, y_j, \ldots, \partial_\alpha y_j) = 0, \quad (1 \le i \le n,\ 1 \le j \le m,\ 1 \le k \le r) \tag{15}$$

one looks for one-parameter infinitesimal transformations

$$\begin{cases} \tilde{x}_i(\lambda) = x_i + \xi_i(x_i, y_j)\lambda + O(\lambda^2), \\ \tilde{y}_j(\lambda) = y_j + \eta_j(x_i, y_j)\lambda + O(\lambda^2), \end{cases} \quad (1 \le i \le n,\ 1 \le j \le m). \tag{16}$$

The conditions of invariance of (15) under transformations (16) are

$$\hat{Z}^{(\alpha)} f_k(x_i, y_j, \ldots, \partial_\alpha y_j)|_{f_s=0} = 0, \quad (1 \le k, s \le r) \tag{17}$$

$$\hat{Z}^{(\alpha)} = \xi_i \partial_{x_i} + \eta_j \partial_{y_j} + \zeta_{j;i} \partial_{y_{j;i}} + \cdots + \zeta_{j;\alpha} \partial_{y_{j;\alpha}}, \tag{18}$$

where $\partial_i y_j$ denoted by $y_{j;i}$, etc.[3] Functions $\zeta_{j;\ldots}$ involved in the differential operator (18) are uniquely computed in terms of functions ξ_i, η_j and their derivatives by means of the recurrence relations

$$\zeta_{j;i} = D_i(\eta_j) - y_{j;q} D_i(\xi_q),$$
$$\zeta_{j;i_1 \ldots i_p} = D_{i_p}(\zeta_{j;i_1 \ldots i_{p-1}}) - y_{j;i_1 \ldots i_{p-1} q} D_{i_p}(\xi_q)\,,$$

where D_i is the total derivative operator with respect to x_i

$$D_i = \partial_i + y_{j;i} \partial_{y_j} + y_{j;ik} \partial_{y_{j;k}} + \cdots$$

The invariance conditions (17) produce the overdetermined system of linear homogeneous PDEs in ξ_i, η_j which is called the *determining system*. Its particular solution yields an infinitesimal operator of the symmetry group

$$\hat{Z} = \xi_i\, \partial_{x_i} + \eta_j\, \partial_{y_j}\,, \tag{19}$$

and the general solution yields all the infinitesimal operators.

Given initial system (15), integration of the determining system is generally a bottleneck of the whole procedure of constructing these symmetry operators, and completion the system to involution is the most universal algorithmic method of its integration [28].

[3] In this section the summation over repeated indices is always assumed.

Example 36. [33] Diffusion type equation $y_t + yy_x - ty_{xx} = 0$ $(n = 2, m = 1)$. The symmetry operator (18) of the form

$$\hat{Z} = \xi_1 \, \partial_t + \xi_2 \, \partial_x + \eta \, \partial_y \tag{20}$$

satisfies the determining system

$$\begin{aligned}
&\partial_{yy}\xi_1 = 0, \quad \partial_{yy}\xi_2 = 0, \quad t\,\partial_{yy}\eta - 2\,t\,\partial_{xy}\xi_2 - 2\,y\,\partial_y\xi_2 = 0,\\
&\partial_y\xi_1 = 0, \quad 2\,t^2\partial_{xy}\eta - t^2\partial_{xx}\xi_2 - y\,t\,\partial_x\xi_2 + t\,\partial_t\xi_2 + y\,\xi_1 - t\,\eta = 0,\\
&t\,\partial_{xx}\eta - y\,\partial_x\eta - \partial_t\eta = 0, \quad t^2\,\partial_{xx}\xi_1 - y\,t\,\partial_x\xi_1 + 2\,t\,\partial_x\xi_2 - t\,\partial_t\xi_1 - \xi_1 = 0,\\
&t\,\partial_{xy}\xi_1 + \partial_y\xi_2 = 0, \quad \partial_x\xi_1 = 0.
\end{aligned}$$

By choosing the orderly degree-reverse-lexicographical ranking with $\partial_y \succ \partial_x \succ \partial_t$, $\xi_1 \succ \xi_2 \succ \eta$ and applying the completion algorithm of Sect. 5, we obtain the (Pommaret, Janet, lexicographically induced) involutive system

$$\begin{aligned}
&\partial_y\xi_1 = 0, \quad \partial_y\xi_2 = 0, \quad \partial_y\eta = 0, \quad \partial_x\xi_1 = 0, \quad \partial_x\xi_2 - \frac{1}{t}\,\xi_1 = 0,\\
&\partial_x\eta = 0, \quad \partial_t\xi_1 - \frac{1}{t}\,\xi_1 = 0, \quad \partial_t\xi_2 - \eta = 0, \quad \partial_t\eta = 0.
\end{aligned}$$

The generators of parametric derivatives ξ_1, ξ_2, η have no multipliers. Hence, the general solution depends on three arbitrary constants c_1, c_2, c_3, and it can easily be obtained by explicit integration of the involutive system

$$\xi_1 = c_1 t, \quad \xi_2 = c_1 x + c_2 t + c_3, \quad \eta = c_2.$$

Respectively, the Lie symmetry group is three-dimensional. Its symmetry operators $\hat{Z}_1 = t\partial_t + x\partial_x$, $\hat{Z}_2 = t\partial_x + \partial_y$, $\hat{Z}_3 = \partial_x$ form the Lie algebra $[\hat{Z}_1, \hat{Z}_2] = 0$, $[\hat{Z}_2, \hat{Z}_3] = 0$, $[\hat{Z}_1, \hat{Z}_3] = -\hat{Z}_3$.

Example 37. [38] The Harry Dym equation $\partial_t y - y^3\partial_{xxx}y = 0$ $(n = 2, m = 1)$ which was already used in [28] as an illustrative example. The symmetry operator in the form (20) is now determined by the system

$$\begin{aligned}
&\partial_y\xi_1 = 0, \quad \partial_x\xi_1 = 0, \quad \partial_y\xi_2 = 0, \quad \partial_{yy}\eta = 0,\\
&\partial_{xy}\eta - \partial_{xx}\xi_2 = 0, \quad \partial_t\eta - y^3\partial_{xxx}\eta = 0,\\
&3\,y^3\partial_{xxy}\eta + \partial_t\xi_2 - y^3\partial_{xxx}\xi_2 = 0, \quad y\,\partial_t\xi_1 - 3\,y\,\partial_x\xi_2 + 3\,\eta = 0.
\end{aligned}$$

Its Janet and Pommaret involutive form for the same ranking as in the previous example is

$$\begin{aligned}
&\partial_{xx}\eta = 0, \quad \partial_{xt}\eta = 0, \quad \partial_y\eta - \frac{1}{y}\,\eta = 0, \quad \partial_t\eta = 0, \quad \partial_y\xi_2 = 0,\\
&\partial_x\xi_2 - \frac{1}{3}\,\partial_t\xi_1 - \frac{1}{y}\,\eta = 0, \quad \partial_t\xi_2 = 0, \quad \partial_{tt}\xi_1 = 0, \quad \partial_y\xi_1 = 0, \quad \partial_x\xi_1 = 0.
\end{aligned}$$

There are five generators of parametric derivatives $\xi_1, \partial_t\xi_1, \xi_2, \eta, \partial_x\eta$ which have no multipliers that implies the five-dimensional Lie symmetry group. The involutive determining system in this example is also easy to integrate:

$$\xi_1 = c_1 + c_2 t, \quad \xi_2 = c_3 + c_4 x + c_5 x^2, \quad \eta = (c_4 - \frac{1}{3}\, c_2 + 2\, c_5 x)\, y.$$

This gives the Lie symmetry operators

$$Z_1 = \partial_t, \;\; Z_2 = t\,\partial_t - \frac{1}{3} y\,\partial_y, \;\; Z_3 = \partial_x, \;\; Z_4 = x\,\partial_x + y\,\partial_y, \;\; Z_5 = x^2\partial_x + 2xy\,\partial_y$$

with the following nonzero commutators of the symmetry algebra

$$[Z_1, Z_2] = Z_1, \quad [Z_3, Z_4] = Z_4, \quad [Z_3, Z_5] = 2\, Z_4, \quad [Z_4, Z_5] = Z_5.$$

8 Conclusion

Most of the above presented definitions, statements and constructive methods can be extended to finite sets of differential polynomials in $\mathbb{R}$ which, given a ranking, are linear with respect to their highest rank (*principal*) derivatives. In Riquier-Janet theory the corresponding systems of PDEs are called *orthonomic*. Their completion to involution, for any constructive and noetherian division, could be done much like linear systems. The essential obstruction here is a non-orthonomic integrability condition. Moreover, even if such an integrability condition is explicitly solvable with respect to its principal derivative, then this leads to non-polynomial orthonomicity, and, thereby, to difficulty in the use of constructive methods of differential and commutative algebra. In the latter case some geometric features of the formal theory may be useful for computational purposes [25].

However, given an orthonomic system of polynomial PDEs and an involutive division L, one can always verify if it is $L-$involutive. Analytic involutive orthonomic systems admit posing an initial value problem providing the existence and uniqueness of solution. One can, hence, determine arbitrariness in the general solution as it is done in Sect. 6 for linear systems. In particular, the compact general formulae (13) and (14) for the Hilbert function and Hilbert polynomial are also valid for involutive orthonomic equations.

We are going to implement the completion algorithm **MinimalLinearInvolutiveBasis** (Sect. 5) after examination and optimization of its polynomial analogue [21,23]. Though its implementation in Reduce for Pommaret division [19] has already shown its efficiency, the differential case needs more careful analysis of implementation and optimization issues to be applicable to PDEs of practical interest. Thus, in Lie symmetry analysis of relatively small systems it is easy to obtain determining systems of many hundreds

equations. Currently, the most efficient completion algorithm for linear systems implemented in some packages for Lie symmetry analysis [28] is that of paper [24]. Its underlying implementations allow to treate hundreds determining equations (cf. [25]). As for significantly larger determining systems, they are hardly tractable by the present day computer algebra tools, whereas there are practical needs in it. In gas dynamics, for instance, the group classification of the system of five second order PDEs describing a viscous heat conducting gas and involving five dependent and four independent variables (three spatial and one temporal) [39], leads to the determining system containing more than 200 000 equations.

In our intention to extract, in the process of implementation, the maximal possible efficiency from the algorithms proposed, we hope, first of all, to detect (heuristically) the most optimal choice of involutive division. As the first step in this direction an implementation of the monomial completion for different divisions has been done in *Mathematica* and for Janet division in C [34].

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